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About the Neuberg–Pedoe and the Oppenheim Inequalities

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This paper is a complete review and unified treatment of recent results conerning the Neuberg–Pedoe and Oppenheim inequalities. Some new proofs and generalizations of these results are also added.

INTRODUCTION

This paper concerns results about the well-known Neuberg–Pedoe and Oppenheim inequalities for the areas of two triangles. Of course, these inequalities were considered in the book "Geometric Inequalities" [1] (hereafter referred to as GI), but after the appearance of this book many new results connected with these inequalities appeared.

First, it was noted that Pedoe's result from GI 10.8 was partly proved in 1891 by J. Neuberg, and many authors call this inequality the Neuberg–Pedoe inequality. Further, A. Oppenheim noted that his inequality from GI 10.12 is equivalent to the Neuberg–Pedoe inequality.

Several papers contain new proofs of the Neuberg–Pedoe inequality. In Section 1 of this paper, we shall show that the idea of a very simple Carlitz proof can be used in the proof of the Oppenheim inequality too.

In Section 2, we give the comments of O. Bottema on the mixed area of two triangles. It is interesting that one of his results is connected with a result of L. Carlitz, which we give in Section 3. An extension of the Carlitz results is also given.

Refinements of the Neuberg–Pedoe inequality were given in several papers. We give some new proofs and some extensions of these results. We also give some similar results for the Oppenheim inequality.

In Section 5 we consider some further generalizations of the Oppenheim inequality for triangles, quadrilaterals, and tetrahedrons. We also give some extensions of these results.

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1. The Neuberg-Pedoe and the Oppenheim Inequality

THEOREM A. *Let a_i, b_i, c_i denote the sides of the triangle $A_iB_iC_i$ ($i=1$, 2) with areas F_i. Then*

$$H \geqslant 16F_1F_2, \tag{1}$$

where $H=\sum a_1^2(-a_2^2+b_2^2+c_2^2)$, with equality if and only if the triangles are similar.

This is the well-known Neuberg–Pedoe inequality (see [2–6] or GI 10.8).

There exist a series of proofs of the Neuberg–Pedoe inequality. For some of these proofs see also [7–11]. L. Carlitz [8] gave a very simple proof of this inequality by using the well-known Aczél inequality, i.e., the following special case of the Aczél inequality:

LEMMA 1. *Let $a=(a_1, ..., a_n)$ and $b=(b_1, ..., b_n)$ be two sequences of real numbers, such that*

$$a_1^2-a_2^2-\cdots-a_n^2>0 \quad \text{and} \quad b_1^2-b_2^2-\cdots-b_n^2>0. \tag{2}$$

Then

$$(a_1^2-a_2^2-\cdots-a_n^2)(b_1^2-b_2^2-\cdots-b_n^2) \leqslant (a_1b_1-a_2b_2-\cdots-a_nb_n)^2 \tag{3}$$

with equality if and only if the sequences a and b are proportional.

A. Oppenheim [12] (see also GI 10.12) gave the following result:

THEOREM B. *Suppose that $A_iB_iC_i$ ($i=1,2$) are triangles with sides a_i, b_i, c_i and areas F_i. Define numbers a_3, b_3, c_3 by $a_3=(a_1^2+a_2^2)^{1/2}$, etc., then a_3, b_3, c_3 are the sides of a triangle with area F_3, and the following inequality is valid:*

$$F_3 \geqslant F_1+F_2, \tag{4}$$

with equality if and only if the triangles are similar.

Remark. Note that a necessary and sufficient condition for triangularity is $F^2>0$.

Now, we shall show that the Oppenheim inequality is a simple consequence of the well-known Bellman inequality [13] (see also [14, p. 38] or [15, p. 58]), i.e., of the following consequence of Bellman's inequality:

LEMMA 2. *If a and b are sequences of nonnegative real numbers which satisfy* (2), *then*

$$\begin{aligned}(a_1^2-a_2^2-\cdots-a_n^2)^{1/2}+(b_1^2-b_2^2-\cdots-b_n^2)^{1/2}\\ \leqslant((a_1+b_1)^2-(a_2+b_2)^2-\cdots-(a_n+b_n)^2)^{1/2}\end{aligned} \tag{5}$$

with equality if and only if the sequences a and b are proportional.

Proof of Theorem B. To prove (4) we take

$$4F_1=((a_1^2+b_1^2+c_1^2)^2-2(a_1^4+b_1^4+c_1^4))^{1/2}$$

and similarly for F_2 and F_3. Now put

$$a_1\to a_1^2+b_1^2+c_1^2,\qquad a_2\to 2^{1/2}a_1^2,\qquad a_3\to 2^{1/2}b_1^2,\qquad a_4\to 2^{1/2}c_1^2,$$

$$b_1\to a_2^2+b_2^2+c_2^2,\qquad b_2\to 2^{1/2}a_2^2,\qquad b_3\to 2^{1/2}b_2^2,\qquad b_4\to 2^{1/2}c_2^2.$$

Then (2) holds and (5) becomes

$$\begin{aligned}4F_1+4F_2&\leqslant\left(\left(\sum a_1^2+\sum a_2^2\right)^2-2\sum(a_1^2+a_2^2)^2\right)^{1/2}\\ &=\left(\left(\sum a_3^2\right)^2-2\sum a_3^4\right)^{1/2}=4F_3\end{aligned}$$

with equality if and only if the triangles are similar.

Remarks. (1) By the same substitution that converts Lemma 2 Theorem B, one also derives Theorem A from Lemma 1.

(2) Oppenheim noted in [16] that Theorems A and B are equivalent. He also noted that Theorem B (therefore also Theorem A) is equivalent to GI 14.1. Analogously, we can easily show that Lemmas 1 and 2 are also equivalent.

(3) Further, it is known that Lemmas 1 and 2 can be easily proved by using the Cauchy and the Minkowski inequalities, respectively (see [17]). Therefore, we can give a simple proof of Theorems A and B by using these inequalities too.

(4) A generalization to several dimensions of the Neuberg–Pedoe inequality is given in [18]. The same is also valid for the Oppenheim inequality (see Section 5 of this paper or [16]). A generalization for two n-gons of the Neuberg–Pedoe inequality is given in [19].

2. Comment by O. Bottema: On the Mixed Area of Two Triangles

2.1. For the area F_1 of the triangle with sides a_1, b_1, c_1 and the angles A_1, B_1, C_1 we have

$$16F_1^2 = -a_1^4 - b_1^4 - c_1^4 + 2b_1^2c_1^2 + 2c_1^2a_1^2 + 2a_1^2b_1^2,$$

a homogeneous quadratic form of a_1^2, b_1^2, c_1^2. For a second triangle with sides a_2, b_2, c_2 the analogous formula holds. We consider the expression

$$16F_{12}^2 = -a_1^2a_2^2 - b_1^2b_2^2 - c_1^2c_2^2 + (b_1^2c_2^2 + b_2^2c_1^2) + (c_1^2a_2^2 + c_2^2a_1^2) + (a_1^2b_2^2 + a_2^2b_1^2), \tag{6}$$

which is the "polar form" of the two quadratics. We shall call F_{12} the "mixed area" of the two triangles. We have

$$8F_{12}^2 = b_1^2c_2^2 + b_2^2c_1^2 - 2b_1c_2b_2c_1 \cos A_1 \cos A_2. \tag{7}$$

Hence $F_{12}^2 > 0$: the mixed area is a real number.

Obviously $F_{12} = F_{21}$, $F_{11} = F_1$, $F_{22} = F_2$.

Furthermore

$$8(F_{12}^2 - F_{11}F_{22}) = b_1^2c_2^2 + b_2^2c_1^2 - 2b_1c_2b_2c_1 \cos (A_1 - A_2) \tag{8}$$

and therefore

$$F_{12}^2 \geqslant F_{11}F_{22},$$

which is the Neuberg–Pedoe inequality.

2.2. Bottema met the concepts in **2.1** by dealing with the following problems. In a plane, two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are given.

Question. Is it possible to place them one to the other in such a way that A_1A_2, B_1B_2, and C_1C_2 are parallel?

Answer. It is possible if and only if

$$2F_{12}^2 \geqslant F_{11}^2 + F_{22}^2. \tag{9}$$

The same condition holds for two triangles in space if one is a parallel projection of the other.

A Second Problem. Once more two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are given. Is it possible to place $A_2B_2C_2$ such that it is an inscribed triangle of $A_1B_1C_1$ (A_2, B_2, C_2 on the corresponding sides—if necessary extended—of $A_1B_1C_1$)?

Answer. The necessary and sufficient condition reads

$$2F_{12}^2 + 2F_1F_2 - F_1^2 \geqslant 0.$$

The above lines are a summary of papers [20, 21].

3. On a Result of L. Carlitz

L. Carlitz [22] proved the following result:

THEOREM C. *If the differences*

$$a_1^2 - a_2^2, \qquad b_1^2 - b_2^2, \qquad c_1^2 - c_2^2 \tag{10}$$

are all positive or all negative and in addition the numbers

$$|a_1^2 - a_2^2|^{1/2}, \qquad |b_1^2 - b_2^2|^{1/2}, \qquad |c_1^2 - c_2^2|^{1/2} \tag{11}$$

form the sides of a triangle Δ (possibly degenerate), then

$$8(F_1^2 + F_2^2) - H = 8F^2(\Delta), \tag{12}$$

where $F(\Delta)$ denotes the area of Δ. Otherwise

$$H \geqslant 8(F_1^2 + F_2^2). \tag{13}$$

Remark. Inequality (13) is the same as (9), so the above result gives more information on triangles satisfying the first Bottema question from Section 2.

Now, we shall give an extension of Theorem C, i.e., the following theorem is valid:

THEOREM D. *If the differences* (10) *are all positive or all negative and in addition the numbers* (11) *form the sides of a triangle Δ (possibly degenerate), then equality* (12) *and equalities*

$$2F_1^2 + 2F_2^2 - F_3^2 = F^2(\Delta), \tag{14}$$

and

$$4F_3^2 - H = 4F^2(\Delta) \tag{15}$$

are valid. Otherwise

$$H > 4F_3^2 > 8(F_1^2 + F_2^2). \tag{16}$$

Proof. The following identity is given in [22]:

$$16(F_1^2+F_2^2)-2H=T, \tag{17}$$

where

$$T=2\sum (a_1^2-a_2^2)(b_1^2-b_2^2)-\sum (a_1^2-a_2^2)^2.$$

Note that the following identity is also valid:

$$16(F_1^2+F_2^2)+2H=16F_3^2. \tag{18}$$

So, we have

$$32(F_1^2+F_2^2)=T+16F_3^2, \tag{19}$$

$$16F_3^2-4H=T. \tag{20}$$

Note that if the differences (10) are all positive or all negative and in addition the numbers (11) form the sides of a triangle Δ, then $T=16F^2(\Delta)$. In this case we have

$$8(F_1^2+F_2^2)\geqslant 4F_3^2\geqslant H. \tag{21}$$

Otherwise $T<0$, so from (19) and (20) we get (16).

Remark. Obviously, (16) is the refinement of (13).

4. Sharpening the Neuberg–Pedoe and the Oppenheim Inequality

K.S. Poh [23] proved the following refinement of (1):

THEOREM E. *Let the conditions of Theorem* A *be satisfied, and let*

$$E=\left(\sum a_1^2\right)\left(\sum a_2^2\right)-2\left(\left(\sum a_1^4\right)\left(\sum a_2^4\right)\right)^{1/2}.$$

Then

$$H\geqslant E\geqslant 16F_1F_2 \tag{22}$$

and $E=16F_1F_2$ *if and only if the two triangles are equibrocardian, i.e., if and only if the two triangles have equal Brocard's angles. Moreover, the following are equivalent*:

(i) $H=16F_1F_2$, (ii) $H=E$, (iii) $\Delta A_1B_1C_1\sim\Delta A_2B_2C_2$.

J. F. Rigby [24] gave a short proof of Theorem C. In the proof of inequality

$$E \geqslant 16F_1F_2 \tag{23}$$

he starts from the fact that $4F_i = \sqrt{X_i^2 - 2Y_i}$, where $X_i = \sum a_i^2$, $Y_i = \sum a_i^4$ $(i = 1, 2)$.

We shall note that (23) is also a simple consequence of Lemma 1. Indeed, we have

$$16F_1F_2 = \sqrt{X_1^2 - 2Y_1}\sqrt{X_2^2 - 2Y_2} \leqslant X_1X_2 - 2\sqrt{Y_1Y_2} = E,$$

where we used (3) for $n = 2$. Equality occurs if and only if $X_1^2/Y_1 = X_2^2/Y_2$. Since [23]

$$\sum a^4 = 8F^2\left(\left(\sum \operatorname{cotg} A\right)^2 - 1\right) \quad \text{and} \quad \sum a^2 = 4F \sum \operatorname{cotg} A,$$

condition for equality is equivalent with

$$\left(\sum \operatorname{cotg} A_1\right)^2 = \left(\sum \operatorname{cotg} A_2\right)^2,$$

i.e.,

$$\sum \operatorname{cotg} A_1 = \sum \operatorname{cotg} A_2 \tag{24}$$

since $\sum \operatorname{cotg} A_i \geqslant \sqrt{3}$ $(i = 1, 2)$ (GI 2.38).

In the formuation of his theorem, Poh gave Eq. (24) for $E = 16F_1F_2$. We remark that $\operatorname{cotg} \omega = \sum \operatorname{cotg} A$, ω being the Brocard angle of the triangle, so (24) becomes $\operatorname{cotg} \omega_1 = \operatorname{cotg} \omega_2$, i.e., $\omega_1 = \omega_2$. Therefore $E = 16F_1F_2$ if and only if the two triangles are equibrocardian.

Further, we shall note that inequalities (22) can be proved by the method of Carlitz, if instead of Lemma 1 we use the following refinement of Aczél's inequality:

LEMMA. 3. *If the conditions of Lemma* 1 *are satisfied then*

$$\begin{aligned}(a_1^2 - a_2^2 - \cdots - a_n^2)(b_1^2 - b_2^2 - \cdots - b_n^2) &\leqslant (a_1b_1 - (a_2^2 + \cdots + a_n^2)^{1/2}(b_2^2 + \cdots + b_n^2)^{1/2})^2 \\ &\leqslant (a_1b_1 - a_2b_2 - \cdots - a_nb_n)^2.\end{aligned} \tag{25}$$

Proof. By substitution,

$$n = 2, \quad a_1 \to a_1, \quad b_1 \to b_1, \quad a_2 \to (a_2^2 + \cdots + a_n^2)^{1/2},$$
$$b_2 \to (b_2^2 + \cdots + b_n^2)^{1/2},$$

from Lemma 1 we get the first inequality, and using the Cauchy inequality we get the second inequality.

Of course, we can give a similar extension of Theorem B:

THEOREM F. *Let the condition of Theorem* B *be satisfied. Then*

$$F_3 \geqslant G \geqslant F_1 + F_2, \tag{26}$$

where $G = \frac{1}{4}((\sum a_3^2)^2 - 2((\sum a_1^4)^{1/2} + (\sum a_2^4)^{1/2})^2)^{1/2}$, *and* $F_1 + F_2 = G$ *if and only if the two triangles are equibrocardian. Moreover, the following are equivalent*:

(i) $F_3 = F_1 + F_2$, (ii) $F_3 = G$, (iii) $\Delta A_1B_1C_1 \sim \Delta A_2B_2C_2$.

The proof is similar to the proof of Theorem E. (We use Lemma 2 instead of Lemma 1 for the second inequality, and Minkowski's inequality instead of Cauchy's in the first inequality.)

We also note that inequalities (26) can be proved if instead of Lemma 2 we use the following result:

LEMMA 4. *If the conditions of Lemma* 2 *are satisfied then*

$$\begin{aligned}(a_1^2 - a_2^2 - \cdots - a_n^2)^{1/2} &+ (b_1^2 - b_2^2 - \cdots - b_n^2)^{1/2} \\ &\leqslant ((a_1 + b_1)^2 - ((a_2^2 + \cdots + a_n^2)^{1/2} + (b_2^2 + \cdots + b_n^2)^{1/2})^2)^{1/2} \\ &\leqslant ((a_1 + b_1)^2 - (a_2 + b_2)^2 - \cdots - (a_n + n_n)^2)^2.\end{aligned}$$

Proof is similar to the proof of Lemma 3.

Chia-Kuei Peng [25] proved the following sharpening of the Neuberg–Pedoe inequality:

THEOREM G. *Let the conditions of Theorem* A *be satisfied. Then*

$$H \geqslant 8\left(\frac{S_2}{S_1} F_1^2 + \frac{S_1}{S_2} F_2^2\right) \geqslant 16 F_1 F_2, \tag{27}$$

where $S_1 = \sum a_1^2$ *and* $S_2 = \sum a_2^2$.

Here, we shall show that the following result is valid:

THEOREM H. *With the same conditions as in the previous theorem,*

$$H \geqslant E \geqslant 8\left(\frac{S_2}{S_1} F_1^2 + \frac{S_1}{S_2} F_2^2\right) \geqslant 16 F_1 F_2. \tag{28}$$

Proof. Of course, we need only prove

$$E \geqslant 8\left(\frac{S_2}{S_1} F_1^2 + \frac{S_1}{S_2} F_2^2\right).$$

We have

$$\begin{aligned} E &= S_1 S_2 - 2\left(\left(\sum a_1^4\right)\left(\sum a_2^4\right)\right)^{1/2} \\ &= S_1 S_2 \left(1 - 2\left(\left(\frac{1}{S_1^2}\sum a_1^4\right)\left(\frac{1}{S_2^2}\sum a_2^4\right)\right)^{1/2}\right) \\ &\geqslant S_1 S_2 \left(1 - \frac{1}{S_1^2}\sum a_1^4 - \frac{1}{S_2^2}\sum a_2^4\right) \\ &= 8 S_1 S_2 \left(\frac{F_1^2}{S_1^2} + \frac{F_2^2}{S_2^2}\right), \end{aligned}$$

where we used the arithmetic-geometric mean inequality and the formula

$$16F^2 = S^2 - 2\sum a^4, \qquad \text{i.e., } \sum a^4 = \tfrac{1}{2}S^2 - 8F^2.$$

Gao Ling [26] gave two refinements of the Neuberg–Pedoe inequality. Here we shall give some extensions of his results.

THEOREM I. *Let $A_i B_i C_i D_i$ $(i = 1, 2)$ be two quadrilaterals inscribed in circles, let $B_i C_i = a_i$, $C_i D_i = b_i$, $D_i A_i = c_i$, and $A_i B_i = d_i$ $(i = 1, 2)$, and let F_i denote the areas of $A_i B_i C_i D_i$ $(i = 1, 2)$. If*

$$K = 4(b_1 c_1 + d_1 a_1)(b_2 c_2 + d_2 a_2) - (a_1^2 - b_1^2 - c_1^2 + d_1^2)(a_2^2 - b_2^2 - c_2^2 + d_2^2),$$

then

$$0 \leqslant K - 16 F_1 F_2 < 8(b_1 c_1 + d_1 a_1)(b_2 c_2 + d_2 a_2) \tag{29}$$

with equality if and only if corresponding angles B_1 and B_2 are equal.

Proof. Since angles B_1 and D_1 are supplementary, we have

$$2F_1 = (b_1 c_1 + d_1 a_1) \sin B_1.$$

On the other hand, from the cosine law

$$\overline{A_1 C_1^2} = b_1^2 + c_1^2 + 2 b_1 c_1 \cos B_1 = d_1^2 + a_1^2 - 2 d_1 a_1 \cos B_1$$

and hence

$$a_1^2 - b_1^2 - c_1^2 + d_1^2 = 2(b_1 c_1 + d_1 a_1) \cos B_1.$$

Using these equalities and similar equalities for the quadrilateral $A_2B_2C_2D_2$, we obtain

$$K - 16F_1F_2 = 4(b_1c_1 + d_1a_1)(b_2c_2 + d_2a_2)(1 - \cos(B_1 - B_2))$$

and (29) follows immediately since

$$-1 < \cos(B_1 - B_2) \leqslant 1$$

with equality if and only if $B_1 = B_2$.

Note that the following result is a simple consequence of Theorem I:

THEOREM J. (i) *Let the conditions of Theorem* A *be satisfied. Then*

$$2(b_1c_2 - b_2c_1)^2 \leqslant H - 16F_1F_2 < 2(b_1c_2 + b_2c_1)^2 \tag{30}$$

with equality if and only if $A_1 = A_2$, *and*

$$\tfrac{2}{3}\sum (b_1c_2 - b_2c_1)^2 \leqslant H - 16F_1F_2 < \tfrac{2}{3}\sum (b_1c_2 + b_2c_1)^2$$

with equality if and only if the two triangles are similar.

(ii) *Let the conditions of Theorem* B *be satisfied. Then*

$$\tfrac{1}{4}(b_1c_2 - b_2c_1)^2 \leqslant F_3^2 - (F_1 + F_2)^2 < \tfrac{1}{4}(b_1c_2 + b_2c_1)^2$$

with equality if and only if $A_1 = A_2$, *and*

$$\tfrac{1}{12}\sum (b_1c_2 - b_2c_1)^2 \leqslant F_3^2 - (F_1 + F_2)^2 < \tfrac{1}{12}\sum (b_1c_2 + b_2c_1)^2$$

with equality if and only if the two triangles are similar.

Remarks. (1) Gao Ling [26] proved only the first inequalities in (29) and (30). The above proof is only a simple extension of his proof.

(2) Now, we shall note that (30) follows directly from Bottema's identity (8), i.e.,

$$H - 16F_1F_2 = 2(b_1^2c_2^2 + b_2^2c_1^2) - 4b_1c_2b_2c_1\cos(A_1 - A_2)$$

since $-1 < \cos(A_1 - A_2) \leqslant 1$.

5. Further Generalizations of the Oppenheim Inequality

In this Section we shall give some generalizations of the Oppenheim inequality for triangles, quadrilaterals, and tetrahedra.

First, we shall give the following Oppenheim generalization of his Theorem B:

THEOREM K. *Suppose that* $A_iB_iC_i$ $(i=1, 2)$ *are two triangles. Define for any* $p \geqslant 1$, $a=(a_1^p+a_2^p)^{1/p}$, *etc. Then* a, b, c *are the sides of a triangle. The three areas are connected by the inequality* (*if* $p=1$ *or* 2 *or* 4)

$$F^{p/2} \geqslant F_1^{p/2}+F_2^{p/2} \tag{31}$$

with equality if and only if the triangles are similar.

Oppenheim [16] also showed that the inequality does not hold for $p>4$, and he also gave the conjecture that Theorem K holds for $1 \leqslant p \leqslant 4$.

The case $p=2$ is Theorem B. Note that a generalization of this case for n triangles was given in [27], and one result similar to Theorem K was given in [28, p. 39]. The case $p=1$ is again given elsewhere (*Math. Mag.* **56** (1983), 19; *Amer. Math. Monthly* **90** (1983), 522–523). But, one similar result for n triangles was first given by M. S. Klamkin in [27]. The proof from the *Monthly* is similar to Klamkin's proof. Note that Oppenheim's proof is simpler. He used another inequality of Minkowski (see [29, p. 88] or [14, p. 26]).

A. Oppenheim [30] also proved the following result:

THEOREM L. *Suppose that* $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ *are two inscribable quadrilaterals of sides* $a_1, ..., d_1$ *and* $a_2, ..., d_2$. *Define* a, *etc., by*

$$a=(a_1^p+a_2^p)^{1/p}, \qquad etc. \quad (p \geqslant 1).$$

Then there is an inscribable quadrilateral of sides a, *etc.; the areas satisfy* (*for* $p=1, 2, 4$) *the inequality* (31) *with equality if and only if the given polygons are similar.*

The Oppenheim conjectures that Theorem K and L are also valid for $1 \leqslant p \leqslant 4$ were proved by C. E. Carroll [31], i.e., the following results are valid:

THEOREM M. *Let the conditions of Theorem* K *be fulfilled. Inequality* (31) *is valid in the case when* $1 \leqslant p \leqslant 4$, *too. Apart from trivial cases with* $p=1$ *and* $F_1=F_2=0$, *equality holds if and only if*

$$a_1/a_2=b_1/b_2=c_1/c_2. \tag{32}$$

THEOREM N. *If* $p \geqslant 1$, *if the triangles having areas* F_1 *and* F_2 *are acute or right triangles, and if* a, b, c, F *are as in Theorem* K, *then* (31) *holds with equality if and only if* (32) *holds.*

THEOREM O. *If* $1 \leqslant p \leqslant 4$ *and if two quadrilaterals have sides* $a_1, ..., d_1$ *and* $a_2, ..., d_2$, *then* $a = (a_1^p + a_2^p)^{1/p}$, *etc., are the sides of a quadrilateral, and the maximum areas satisfy* (31). *Equality holds iff the sets* $a_1, ..., d_1$ *and* $a_2, ..., d_2$ *are proportional; but there are trivial exceptions with* $p = 1$ *and* $F_1 = F_2 = 0$.

Now, we shall show that we can give generalizations of these results for the case of n triangles or quadrilaterals. In our results, trivial equality cases are not included.

THEOREM P. *Suppose that* $A_iB_iC_i$ $(i = 1, ..., n)$ *are n triangles. Define for any* $p \geqslant 1$

$$a = \left(\sum_{i=1}^{n} w_i a_i^p\right)^{1/p}, \qquad etc.,$$

where $w_i \geqslant 0$ $(i = 1, ..., n)$. *Then a, b, c are the sides of a triangle. If* $1 \leqslant p \leqslant 4$, *the* $n+1$ *areas are connected by the inequality*

$$F^{p/2} \geqslant \sum_{i=1}^{n} w_i F_i^{p/2} \tag{33}$$

with equality if and only if the given n triangles are similar.

Proof. In the case $w_i = 1$ $(i = 1, ..., n)$ we shall use the mathematical induction method. Indeed, for $n = 2$, this is Theorem M. Suppose that Theorem P for $w_i = 1$ $(i = 1, ..., n)$ *is true for* $n - 1$ *and put*

$$a'_i \to a_i \ (i = 1, ..., n-2), \qquad a'_{n-1} \to (a_{n-1}^p + a_n^p)^{1/p}, \text{ etc.}$$

Then from $(F')^{p/2} \geqslant \sum_{i=1}^{n-1} (F'_i)^{p/2}$ follows $F^{p/2} \geqslant \sum_{i=1}^{n} F_i^{p/2}$, because Theorem M gives $(F'_{n-1})^{p/2} \geqslant F_{n-1}^{p/2} + F_n^{p/2}$. Further, by substitution $a_i \to w_i^{1/p} a_i$, etc. $(i = 1, ..., n)$, we get our result.

Similarly, we can prove the following two theorems:

THEOREM Q. *In the previous theorem let* $p \geqslant 1$, *and let the given n triangles be acute or right triangles. Then* (33) *holds with equality if and only if the given n triangles are similar.*

THEOREM R. *If* $1 \leqslant p \leqslant 4$ *and if n quadrilaterals have sides* a_i, b_i, c_i, d_i $(i = 1, ..., n)$, *then* $a = (\sum_{i=1}^{n} w_i a_i^p)^{1/p}$, *etc., are the sides of a quadrilateral, and the maximum areas satisfy* (33). *Equality holds if and only if the sets* a_i, b_i, c_i, d_i $(i = 1, ..., n)$ *are proportional.*

Remark. It is known that, if a quadrilateral has sides of fixed length, the area is maximum when the vertices lie on a circle.

Now, we shall give two applications of Theorems P and R.

COROLLARY 1. *Of all triangles with the same perimeter, the equilateral triangle has the greatest area.*

Proof. Put $\sum_{i=1}^{n} w_i = 1$, then from (33) for $p = 1$ it follows that the symmetric function $F^{1/2}$ is concave. Therefore $F^{1/2}$ is a Schur-concave function (see [32, p. 64]), and the same is valid for F (see [32, p. 61]), a known result [32, p. 209]. A simple consequence of this result is

$$F\left(\tfrac{1}{3}\sum a, \tfrac{1}{3}\sum a, \tfrac{1}{3}\sum a\right) \geq F(a, b, c),$$

i.e., Corollary 1.

Similarly, we can prove (see, for example, [32, p. 209]):

COROLLARY 2. *Of all quadrilaterals with a given perimeter, the square has the greatest area.*

As we said in Section 1, A. Oppenheim noted that the generalization of Theorem B in several dimensions is also valid. As an example, he gave the following result:

THEOREM S. *If tetrahedra T, T_1, T_2 have edges connected by the equations $a^2 = a_1^2 + a_2^2$, etc., then their volumes satisfy the inequality*

$$V^{2/3} \geq V_1^{2/3} + V_2^{2/3}$$

with equality if and only if T_1 and T_2 are similar.

Similarly to the proof of Theorem P we can prove:

THEOREM T. *If tetrahedra T, T_i $(i = 1, ..., n)$ have edges connected by the equations $a^2 = \sum_{i=1}^{n} w_i a_i^2$, etc., where w_i $(i = 1, ..., n)$ are positive numbers, then their volumes satisfy the inequalities*

$$V^{2/3} \geq \sum_{i=1}^{n} w_i V_i^{2/3}$$

with equality if and only if the T_i $(i = 1, ..., n)$ are similar.

Remark. The method from the proof of Theorem P can be used for generalization of some other similar results:

(1) Let the conditions of Theorem P be satisfied for $p = 2$. Then

$$h^2 \geq \sum_{i=1}^{n} w_i h_i^2, \tag{34}$$

where h_i is the altitude of the triangle $A_iB_iC_i$ $(i=1, ..., n)$, and h is the altitude of the triangle with sides a, b, c, with equality only for similar triangles.

(2) Let the conditions from (1) be satisfied, then analogous results for the circumradii are valid,

$$R^2 \leqslant \sum_{i=1}^{n} w_i R_i^2$$

with equality either if the given triangles are similar or if the triangles are right angled, with vertices corresponding.

(3) Let the conditions of Theorem T be satisfied, then the corresponding altitudes satisfy (34).

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